APPENDIX

A Basic Principles of Counting

In order to find the probability of many events, it is necessary to determine the number of possible outcomes for the experiment involved. This requires us to enumerate (obtain a "count" of) the possibilities. This "count" can be obtained by using one of two methods: (1) list all the possibilities and then proceed to count them (1, 2, 3, . . .); or (2) since it is often not necessary to delineate (obtain a representation of) all possibilities, the count can be determined by calculating its numerical value. In this section, we are going to learn three commonly used methods for obtaining the count by calculation: the fundamental technique and two specific techniques.

Illustration A.1

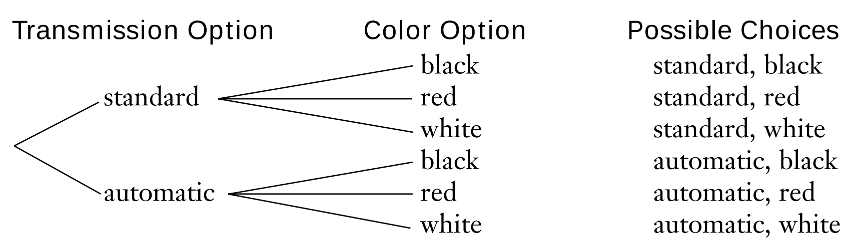
An automobile dealer offers one of its small sporty models with two transmission options (standard or automatic) and in one of three colors (black, red, or white). How many different choices of transmission and color combinations are there for the customer?

SOLUTION

The number of choices available can easily be found by listing and counting them. There are six.

Standard, black	Automatic, black
Standard, red	Automatic, red
Standard, white	Automatic, white

The possible choices can also be demonstrated by use of a tree diagram.



NOTE More information and additional illustrations of tree diagrams can be found in Chapter 4 and in the *Statistical Tutor*.

Each of the two transmission choices can be paired with any one of three colors; thus there are 2×3 or six different possible choices. This suggests the following rule:

FUNDAMENTAL COUNTING RULE

If an experiment is composed of two trials, where one of the trials (single action or choice) has m possible outcomes (results) and the other trial has n possible outcomes, then when the two trials are performed together, there are

$$m \times n \tag{A.1}$$

possible outcomes for the experiment.

In Illustration A.1, $m = 2$ (the number of transmission choices) and $n = 3$ (the number of color choices). Using the Fundamental Counting Rule (formula A.1), the number of possible choices available to a customer is

$$m \times n = 2 \times 3 = 6$$ ■

This fundamental counting rule may be extended to include experiments that have more than two trials.

GENERAL COUNTING RULE

If an experiment is composed of k trials performed in a definite order, where the first trial has n_1 possible outcomes, the second trial has n_2 possible outcomes, the third trial has n_3 outcomes, and so on, then the number of possible outcomes for the experiment is

$$n_1 \times n_2 \times n_3 \times \cdots \times n_k \tag{A.2}$$

Illustration A.2

In many states, automobile license plates use three letters followed by three numerals to make up the "license plate number." (There are other combinations of letters and numerals used; however, let's focus only on this six-character "number" for now.) If we assume that any one of the 26 letters may be used for each of the first three characters and that any one of the 10 numerals 0 through 9 can be used for each of the last three characters, how many different license plate numbers are possible?

SOLUTION

There are 26 possible choices for the first letter ($n_1 = 26$), 26 possible choices for the second letter ($n_2 = 26$), and 26 possible choices for the third letter ($n_3 = 26$). In similar fashion, there are 10 choices for the numeral to be used for each of the fourth ($n_4 = 10$), fifth ($n_5 = 10$), and sixth ($n_6 = 10$) characters. Therefore, using the General Counting Rule [formula (A.2)] we find there are

$$26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17{,}576{,}000$$

different "license plate numbers" using this six-character scheme. ■

Illustration A.3

How many different "license plate numbers" are possible if the nonzero numerals are used for the three leading characters, letters are used for the three trailing characters, and the letters are not allowed to repeat?

SOLUTION

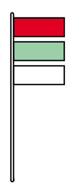
There are 9 possible choices for each of the first three characters (since only 1 through 9 may be used). Thus, $n_1 = 9$, $n_2 = 9$, and $n_3 = 9$. The fourth character may be chosen from any one of the 26 letters ($n_4 = 26$). However, the fifth character must be chosen from any one of the 25 letters not previously used ($n_5 = 25$), and the sixth character must be chosen from the 24 letters not previously used ($n_6 = 24$). Applying the General Counting Rule (A.2), we find there are

$$9 \times 9 \times 9 \times 26 \times 25 \times 24 = 11{,}372{,}400$$

different "license plate numbers" using this second six-character scheme. ■

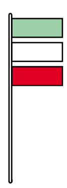
We are now ready to investigate two additional concepts commonly encountered when enumerating possibilities: *permutations* and *combinations*. A permutation is a collection of distinct objects arranged in a specific order, while a combination is a collection of distinct objects without any specific order.

Illustration A.4



There are four flags of different colors (one each of red, white, blue, and green) in a box, and you are asked to select any three. If you select {red, white, green} you have the same combination of colors as {green, red, white}. This question does not require or distinguish between different "orders" or arrangements; thus, each set of flags is one combination. ■

Illustration A.5



There are four flags of different colors (one each of red, white, blue, and green) in a box, and you are asked to select any three of them and make a "signal" by hanging the three different flags, one above the other, on a flagpole. Since red over green over white is different from green over white over red, order is important and each possible signal is one permutation. ■

Permutations

Illustration A.6

Select four different letters from the English alphabet and arrange them in any specific order. As a result of following these instructions, Barbara created the "four-letter word" BSJT. Rob created the word EOST. Steve selected KOCM. How many different "four-letter words" can be created?

Each of these "words" is a *permutation* of four letters selected from the set of 26 different letters forming the alphabet.

PERMUTATION

An ordered arrangement of a set of distinct objects. That is, there is a first object, a second object, a third object, and so on; and each object is distinctly different from the others.

The number of permutations that can be formed is calculated using an adaptation of the General Counting Rule.

PERMUTATION FORMULA

The number of permutations that can be formed using *r* different objects selected from a set of *n* distinct objects (symbolized by ${}_nP_r$ and read "the number of permutations of *n* objects selected *r* at a time") is

$$ {}_nP_r = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1) \tag{A.3}$$

or, in factorial notation,

$$ {}_nP_r = \frac{n!}{(n-r)!} \tag{A.4}$$

NOTE More information and additional illustrations of factorial notation can be found in Chapter 5 and in the *Statistical Tutor*. Remember: 0! = 1.

Let's continue with the solution of Illustration A.6. Since the 4 letters were selected from the 26 letters of the alphabet, $r = 4$ (the number of selections) and $n = 26$ (the number of objects available for selection). Using formula (A.3),

$${}_nP_r = n \times (n-1) \times \cdots \times (n-r+1):$$

$$\begin{aligned} {}_{26}P_4 &= 26 \times 25 \times \cdots \times (26-4+1) \\ &= 26 \times 25 \times 24 \times 23 = 358{,}800 \end{aligned}$$

Or using formula (A.4),

$${}_nP_r = \frac{n!}{(n-r)!}:$$

$$\begin{aligned} {}_{26}P_4 &= \frac{26!}{(26-4)!} = \frac{26!}{22!} \\ &= \frac{26 \times 25 \times 24 \times 23 \times 22 \times 21 \times \cdots \times 1}{22 \times 21 \times \cdots \times 1} \\ &= \frac{26 \times 25 \times 24 \times 23 \times (22 \times 21 \times \cdots \times 1)}{(22 \times 21 \times \cdots \times 1)} \\ &= 26 \times 25 \times 24 \times 23 = 358{,}800 \end{aligned}$$

Illustration A.7

A group of eight finalists in a ceramic art competition are to be awarded five prizes—first, second, and so on. How many different ways are there to award these five prizes?

SOLUTION

Since the prizes are ordered, this is a permutation of $n = 8$ different people, taken 5 at a time (only five prizes, and each prize is distinctly different from the others). Using formula (A.3), we find there are

$${}_nP_r = n \times (n-1) \times \cdots \times (n-r+1):$$

$$\begin{aligned} {}_8P_5 &= 8 \times 7 \times \cdots \times (8-5+1) \\ &= 8 \times 7 \times 6 \times 5 \times 4 = 6{,}720 \end{aligned}$$

different possible ways of awarding these five prizes. ■

Combinations

Illustration A.8

Select a set of four different letters from the English alphabet. As a result of following this instruction, Kevin selected A, E, R, and T. Karen selected D, E, N, and Q. Sue selected R, E, A, and T. Notice that Kevin and Sue selected the same set of letters, even though they selected them in different orders. These three people have selected two different sets of four letters. How many different sets of four letters can be selected?

SOLUTION

Each of these "sets" of four letters represents a *combination* of $r = 4$ objects having been selected from a set of $n = 26$ distinct objects.

COMBINATION

A set of distinct objects without regard to an arrangement or an order. That is, the membership of the set is all that matters.

The number of combinations that can be selected is related to the number of permutations. In Illustration A.6, we found that there were 358,800 permutations of four letters possible. Many permutations were "words" formed from the same set of four letters. For example, the set of four letters A, B, C, and D can be used to form many permutations ("words"):

ABCD	ABDC	ACBD	ACDB	ADBC	ADCB
BACD	BADC	BCAD	BCDA	BDAC	BDCA
CBAD	CBDA	CABD	CADB	CDBA	CDAB
DBCA	DBAC	DCBA	DCAB	DABC	DACB

There are 4! ($4 \times 3 \times 2 \times 1$) or 24 different permutations for this set of four letters. Every other set of four letters can also be used to form 24 permutations. Therefore, if we divide the number of permutations possible (358,800) by the number of permutations each set has (24), the quotient will be the number of different sets (combinations) possible. That is, there are 14,950 (358,800/24) combinations of four letters possible. This concept is generalized in the following formula:

COMBINATION FORMULA

The number of combinations of r objects that can be selected from a set of n distinct objects (symbolized by ${}_nC_r$, and read "the number of combinations of n things taken r at a time") is

$${}_nC_r = \frac{n(n-1)(n-2) \times \cdots \times (n-r+1)}{r!} \tag{A.5}$$

or, in factorial notation,

$${}_nC_r = \frac{n!}{(n-r)! \times r!} \tag{A.6}$$

Let's continue with the solution of Illustration A.8 using these new formulas. First using formula (A.5),

$${}_nC_r = \frac{n(n-1)(n-2)\times\cdots\times(n-r+1)}{r!}:$$

$${}_{26}C_4 = \frac{26(25)(24)\times\cdots\times(26-4+1)}{4!} = \frac{26\times25\times24\times23}{4\times3\times2\times1}$$

$$= \frac{358{,}800}{24} = 14{,}950$$

Using formula (A.6),

$${}_nC_r = \frac{n!}{(n-r)!\times r!}:$$

$${}_{26}C_4 = \frac{26!}{(26-4)!\times4!} = \frac{26!}{22!\times4!}$$

$$= \frac{26\times25\times24\times23\times22\times21\times\cdots\times2\times1}{(22\times21\times\cdots\times2\times1)(4\times3\times2\times1)}$$

$$= \frac{26\times25\times24\times23\times(22\times21\times\cdots\times2\times1)}{(22\times21\times\ldots\times2\times1)(4\times3\times2\times1)}$$

$$= \frac{358{,}800}{24} = 14{,}950$$

■

Illustration A.9

A department has 30 members and a committee of 5 people is needed to carry out a task. How many different possible committees are there?

SOLUTION

As stated, there is no specific assignment or order to the members of the committee; therefore, each possible committee is a combination and $n = 30$ (the number of people eligible to be selected), and $r = 5$ (the number to be selected).

$${}_nC_r = \frac{n!}{(n-r)!\times r!}:$$

$${}_{30}C_5 = \frac{30!}{(30-5)!\times5!} = \frac{30!}{25!\times5!}$$

$$= \frac{30\times29\times28\times27\times26\times25\times24\times\cdots\times2\times1}{(25\times24\times\cdots\times2\times1)(5\times4\times3\times2\times1)}$$

$$= \frac{30\times29\times28\times27\times26\times(25\times24\times\ldots\times2\times1)}{(25\times24\times\ldots\times2\times1)(5\times4\times3\times2\times1)}$$

$$= \frac{17{,}100{,}720}{120} = 142{,}506$$

It is possible to select 142,506 different committees of 5 people from this department of 30 people. ■

NOTE The number of combinations ${}_nC_r$ and the binomial coefficient $\binom{n}{r}$ or $\binom{n}{x}$ are numerically equivalent.

The three "counting" formulas described above in this section [formulas (A.2), (A.4), and (A.6)] can be and often are used together to solve problems.

Illustration A.10

A department has 30 members and a committee is needed to carry out a task. The committee is to be composed of a chairperson and four members. How many different possible committees are there?

SOLUTION

This problem is solved by treating it in two parts: consider the chairperson position and the committee members are two separate parts to be combined using the Fundamental Counting Rule ($m \times n$). Let m be the number of possible choices for the chairperson. Since any one of the 30 department members could serve as the chair, $m = 30$. Let n be the number of four-person committees that can be selected from the remaining 29 department members. Since these four have no specific assignment, the number of possibilities, n, is the number of combinations of 29 things taken 4 at a time.

$$\begin{aligned} n(\text{committees}) &= m \times n \\ &= 30 \times {}_{29}C_4 \\ &= 30 \times \frac{29 \times 28 \times 27 \times 26 \times (25 \times 24 \times \cdots \times 2 \times 1)}{(25 \times 24 \times \cdots \times 2 \times 1) \times 4 \times 3 \times 2 \times 1} \\ &= 30 \times 23{,}751 = 712{,}530 \end{aligned}$$

712,530 different committees of size 5 with an assigned chair are possible.

Illustration A.11

A department has 30 members and a committee is needed to carry out a task. The committee is to be composed of two co-chairpersons and three members. How many different possible committees are there?

SOLUTION

This problem is solved by treating it in two parts: consider the selecting of two co-chairpersons and then the remaining committee members as two separate parts to be combined using the Fundamental Counting Rule ($m \times n$). Let m be the number of possible choices for the co-chairpersons. This is like a committee of two, since there is no further distinction between them; therefore $m = {}_{30}C_2$, since any two of the 30 department members could serve as the co-chairs. Let n be the number of three-person committees that can be selected from the remaining 28 department members. Since these three have no specific assignment, the number of possibilities, n, is the number of combinations of 28 things taken 3 at a time.

$$\begin{aligned} n(\text{committees}) &= m \times n \\ &= {}_{30}C_2 \times {}_{28}C_3 \\ &= \frac{30!}{28! \times 2!} \times \frac{28!}{25! \times 3!} \\ &= \frac{(30 \times 29 \times 28 \times 27 \times 26 \times 25 \times \cdots \times 1) \times (28 \times 27 \times \cdots \times 1)}{(28 \times 27 \times \cdots \times 1) \times (2 \times 1) \times (25 \times \cdots \times 1) \times (3 \times 2 \times 1)} \\ &= 15 \times 29 \times 28 \times 9 \times 13 \\ &= 1{,}425{,}060 \end{aligned}$$

1,425,060 different committees of size 5 with assigned co-chairpersons are possible. ■

EXERCISES

A.1 A long weekend of three days is being planned. The three days are to be spent taking scenic drives through the countryside and ending each day at a motel where reservations have been previously made for Friday and Saturday nights (will be home Sunday night). There are three scenic routes that may be traveled on Friday, two choices for Saturday, and three scenic route choices for Sunday's return trip. How many different trips are possible if
a. all the scenic options are considered?
b. one of the Friday routes has been previously driven, and is not a choice?
c. on Sunday, it is decided to drive straight home and not take a scenic route?

A.2
a. Show that formulas (A.3) and (A.4) are equivalent.
b. Show that formulas (A.5) and (A.6) are equivalent.

A.3 Explain why each of the following pairs of "counts" are equal:
a. ${}_nP_n$ and ${}_nP_{n-1}$
b. ${}_nP_1$ and ${}_nC_1$
c. ${}_nC_r$ and ${}_nC_{n-r}$
d. ${}_nC_r$ and the binomial coefficient $\binom{n}{r}$

A.4 A department of 30 people is to select a committee of 5 persons. How many different committees are possible if the committee is composed of
a. a chairperson, a secretary, and three others?
b. two co-chairs and three others?
c. two co-chairs, a secretary, and two others?

A.5 License plates are to be "numbered" using a combination of letters and numerals. How many different "numbers" are possible if each of the following sets of restrictions is used?
a. Six characters using any combination or arrangement of the 26 letters and 10 single-digit numerals.
b. Six characters using any combination or arrangement of letters and single-digit numerals, except that "zero" and "one" are not to be used because they are hard to distinguish from "o" and "i."
c. Six characters using letters for the two leading characters and the 10 single-digit numerals for the four trailing characters.
d. Six characters using the 10 single-digit numerals for the four leading characters and letters for the two trailing characters.
e. Six characters using the 10 single-digit numerals for the four leading characters and letters for the two trailing characters, except that "zero" cannot be the leading character.

A.6 Mathew has six shirts, four pairs of pants, and five pairs of socks clean and ready to wear. How many different "outfits" can he assemble if
a. he wears one item from each category?
b. he wears one specific shirt and one item from the other two categories?
c. he only wears two of the shirts with one specific pair of pants and no socks, but the rest are worn in any complete combination?

A.7 Five cards are to be randomly selected from a standard bridge deck of 52 cards.
a. How many different "hands" of five cards are possible?
b. How many different "hands" of five cards are possible if the first drawn is an ace?
c. How many different "hands" of five cards are possible if the first card drawn is an ace and the remaining four are not aces?
d. How many different "hands" of five cards are possible if the first card drawn is a club and the remaining four are not clubs?
e. How many different "hands" of five cards are possible if the first card drawn is an ace and the remaining four are not clubs?

A.8 Millions of people play the large lotteries; some play regularly while others play only when the prize is very large. Powerball is one of the largest and is played in many states. A ticket costs $1 and the players choose five numbers from 1 to 49 and a Powerball from 1 to 42.
a. Use combinations to calculate the exact odds that someone will match all six numbers.
b. Does your answer (a) verify the 1 in 80.1 million odds advertised by the game?

Appendix A Answers

A.1 a. 18 b. 12 c. 6

A.3 a. ${}_nP_n = n!$ and ${}_nP_{n-1} = n \times n - 1 \times n - 2 \times \cdots \times 2$. ${}_nP_{n-1}$ is the same product as ${}_nP_n$, only the last factor of 1 is missing.

b. Both have value n.

c. ${}_nC_r = \dfrac{n!}{r!(n-r)!}$ and

$${}_nC_{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!}$$

d. They are two different names and applications for the same techniques.

A.5 a. 2,176,782,336 assuming zero and 0 can be used as first characters.

b. 1,544,804,416, assuming "0" can be the first character.

c. 6,760,000 d. 6,760,000 e. 6,084,000

A.7 a. ${}_{32}C_5 = 2{,}598{,}960$

b. ${}_4C_1 \times {}_{31}C_4 = 4 \times 249{,}900 = 999{,}600$

c. ${}_4C_1 \times {}_{48}C_4 = 4 \times 194{,}580 = 778{,}320$

d. ${}_{13}C_1 \times {}_{30}C_4 = 13 \times 82{,}251 = 1{,}069{,}263$

e. ${}_1C_1 \times {}_{39}C_4 + {}_3C_1 \times {}_{38}C_4 = 303{,}696$

A.8 a. ${}_{48}C_5 \times {}_{42}C_1 = 1{,}906{,}884 \times 42 = 80{,}089{,}128$

b. Yes, one chance in 80.1 million